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TRANSCENDENTAL DIVISION ALGEBRAS AND SIMPLE NOETHERIAN RINGS

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ABSTRACT

In this paper we prove the following theorem:

Let D be a division ring with center the field k, and let $k(x_1, \cdots, x_n)$ denote the rational function field in n variables over k. If D contains a maximal subfield which has transcendence degree at least n over k, then $D \otimes_k k(x_1, \cdots, x_n)$ is a simple Noetherian domain of Krull and global dimensions n.

Rather surprisingly, the preceding result can be used to determine the maximum transcendence degrees of the commutative subalgebras of several classically studied division rings. Using the theorem we prove, for example, that in the division ring of quotients of the Weyl algebra, A_n, every maximal subfield has transcendence degree at most n over the center.

1. Introduction

Let D be a division ring with center the field k. Adopting the usual definition for field extensions, we say that D is transcendental over k if D is not an algebraic k-algebra. While examples of transcendental division algebras have been known since the end of the last century, there are relatively few theorems about the general properties of these rings or about their internal structure. Of these, perhaps the best known is an old result which asserts that if D is transcendental over k then the polynomial ring $D[x]$ is primitive. This in turn implies that the tensor product $D \otimes_k k(x)$ is a simple Noetherian domain, not a division ring. This classical theorem served as the chief motivation for the results of this paper. We first prove that when D is transcendental $D \otimes k(x_1, \cdots, x_n)$ is simple Noetherian, never Artinian, for every integer n. We then consider the Krull and global dimensions of the ring $D \otimes k(x_1, \cdots, x_n)$ and show that these properties depend strongly on the transcendence degrees of the maximal subfields of D.

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We turn now to a brief description of the plan of the paper. As each section has its own introduction, we discuss here only the main points and their organization. Section 2 introduces r-sequences, or generalized R-sequences, which are our principal machinery. The results of this section are mainly technical, but are a necessary preliminary to the later developments. In Section 3 we define the transcendence degree of a k-algebra and study the relation of this concept to the structure of the ring $A(x_1,\cdots,x_n)=A\otimes_k k(x_1,\cdots,x_n)$. The proof of the main theorem is based on these more general considerations. Section 4 is devoted completely to examples. There we show not only that some hypothesis on transcendence degree is necessary in the theorem, but also that it can frequently be applied to give exact bounds on the transcendence degrees of the subfields of a division ring.

Finally, a few remarks about the conventions to be observed here. All rings and algebras are associative with unit element. "Module" almost always means right module, and we similarly abbreviate right Noetherian to Noetherian, right Krull dimension to Krull dimension. The term field is used exclusively for commutative fields, but a domain is simply a nonzero ring without zero-divisors and need not be commutative. Additional conventions and notation will be introduced as needed.

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2. Generalized R-sequences

R-sequences play an important role in modern commutative algebra and are especially useful in problems concerning the dimensions of commutative Noetherian rings [13, chapter 3], [15, sections 15, 16, 18]. In this section we introduce regular endomorphism sequences, or *r-sequences*, which provide a generalization of R-sequences to noncommutative rings.

After giving the necessary definitions (2.1, 2.2) and a few examples, we establish those properties of r-sequences which will be needed in Section 3. Theorem 2.5 gives a necessary and sufficient condition for an r-sequence to be preserved under localization. Theorem 2.6 relates r-sequences to Krull dimension [8], and shows that if a ring contains an r-sequence of length n then its Krull dimension is at least n. Theorem 2.8 determines the homological dimension of R/U when U is a right ideal generated by a commuting r-sequence.

Throughout this section R is a ring and M_R, often simply M, a right R-module. $\mathrm{End}_R(M)$ denotes the ring of R-endomorphisms of M. We follow the usual convention of writing morphisms opposite scalars, and thus regard M as an $(\mathrm{End}_R(M), R)$-bimodule.

Definition 2.1. A sequence of endomorphisms $\alpha_1, \cdots, \alpha_n$ in $\mathrm{End}_R(M)$ is called a regular endomorphism sequence on M, for brevity an *r-sequence on M*, if it satisfies the following conditions:

(i) $\alpha_1 M + \cdots + \alpha_n M \neq M$;

(ii) $\alpha_k(\alpha_1 M + \cdots + \alpha_{k-1} M) \subset \alpha_1 M + \cdots + \alpha_{k-1} M$, for $k > 1$;

(iii) $\alpha_1^{-1}(0) = 0$; $\alpha_k^{-1}(\Sigma_{i=1}^{k-1} \alpha_i M) = \Sigma_{i=1}^{k-1} \alpha_i M$, for $k > 1$.

Here αN and $\alpha^{-1} N$ denote respectively the direct and inverse image of the submodule N under the map α.

Definition 2.2. A sequence $u_1, \cdots, u_n$ in the ring R is called an *r-sequence in R* if under the regular representation $R \simeq \mathrm{End}_R(R)$, the images of the u_i form an r-sequence on the right R-module R_R.

Remark 2.3. (i) A monomorphism $0 \to M \xrightarrow{\alpha} M$ is called *proper* if $\mathrm{Im}\, \alpha \neq M$. If $\alpha_1, \cdots, \alpha_n$ is an r-sequence on M, then α_1 is a proper monomorphism of M and for each $k > 1$, α_k induces a proper monomorphism of the factor $M_{k-1} = M/(\alpha_1 M + \cdots + \alpha_{k-1} M)$.

(ii) Let M_{k-1} be as above ($M_0 = M$), and let $\bar{\alpha}_k$ denote the induced map. A useful reformulation of 2.1(iii) is that for each k, $1 \leqq k \leqq n$, the following sequence be exact:

$$0 \to M_{k-1} \xrightarrow{\bar{\alpha}_k} M_{k-1} \to M_k \to 0.$$

(iii) When R is commutative there is for any R-module M a canonical homomorphism $R \xrightarrow{\mu} \mathrm{End}_R(M)$. Explicitly, $\mu(r) = \mu_r$ where $\mu_r(m) = mr$. Via this mapping, one sees that a sequence $u_1, \cdots, u_n$ in R is an r-sequence in the sense of 2.1 iff $u_1, \cdots, u_n$ is an R-sequence as defined in commutative algebra [13, p. 84]. Accordingly, r-sequences are generalized R-sequences. We have introduced the "r" notation to avoid possible confusion with the more traditional notion.

We now give a few examples of r-sequences.

Example 2.4. (i) Let R be any ring. The prototype of all r-sequences is the sequence $x_1, \cdots, x_n$ in the polynomial ring $R[x_1, \cdots, x_n]$. By way of contrast, in the ring of noncommuting polynomials $R\langle x_1, \cdots, x_n \rangle$ the indeterminates do not form an r-sequence.

(ii) Let R be any ring and let $y_1, \cdots, y_n$ in R be a "regular normalizing set" as defined by Walker [23, p. 28]. It is easy to show that such a sequence is in fact an r-sequence, and the same observation applies to a "regular centralizing set".

(iii) Let k be a field of characteristic zero and let $A_n(k)$, or simply A_n, denote the Weyl algebra over k. A_n is the k-algebra with $2n$ generators $p_1, \cdots, p_n$; $q_1, \cdots, q_n$ and relations $[p_i, q_j] = p_iq_j - q_jp_i = \delta_{ij}$, $[p_l, p_m] = 0 = [q_l, q_m]$. As noted by Rinehart [17, p. 243] both $p_1, \cdots, p_n$ and $q_1, \cdots, q_n$ are r-sequences in A_n.

Our first theorem on r-sequences concerns their behavior under classical localization. We consider the following situation: M is an R-module, $S \subset R$ is a right Ore set, R_S and M_S are the ring and module of fractions. For the relevant definitions and constructions see [22, chapter 2]. There is a homomorphism $\mathrm{End}_R(M) \to \mathrm{End}_{R_S}(M_S)$ defined by $\alpha \to \hat{\alpha}$, where $\hat{\alpha}(ms^{-1}) = (\alpha m)s^{-1}$.

THEOREM 2.5. *Let M, R, S be as above. Suppose that $\alpha_1, \cdots, \alpha_n$ is an r-sequence on M with images $\hat{\alpha}_1, \cdots, \hat{\alpha}_n$ in* $\mathrm{End}(M_S)$. *Then $\hat{\alpha}_1, \cdots, \hat{\alpha}_n$ is an r-sequence on M_S iff $\hat{\alpha}_1 M_S + \cdots + \hat{\alpha}_n M_S \neq M_S$.*

PROOF. ($\Leftarrow$) We verify the three conditions of 2.1. We have 2.1(i) by hypothesis. Since $\hat{\alpha}(N_S) = (\alpha N)_S$ for any submodule $N \subset M$ and localization commutes with sums, 2.1(ii) follows. For 2.1(iii), use 2.3(ii) and the fact that localization is exact [22, p. 57].

($\Rightarrow$) This is clear.

We turn to the relation of r-sequences to Krull dimension. For the definition and basic properties of Krull dimension, see [8]. Following [8], we use the notation $\mathrm{K}\dim M$ to denote the Krull dimension of a right R-module M. The next proposition is an easy generalization of the fact that if R has Krull dimension and $a \in R$ is right regular then $\mathrm{K}\dim R/aR < \mathrm{K}\dim R$.

THEOREM 2.6. *Let M be an R-module, $\alpha_1, \cdots, \alpha_n$ an r-sequence on M, and put $\bar{M} = M/(\alpha_1 M + \cdots + \alpha_n M)$. If M has Krull dimension, then* $\mathrm{K}\dim M \geqq \mathrm{K}\dim \bar{M} + n$.

PROOF. Defining the length of any r-sequence to be its cardinality, the proof is by induction on the length. For $n = 1$, we have an R-module M with proper monomorphism α. Easily $\alpha^j M/\alpha^{j+1} M \simeq M/\alpha M$ for all j. This implies that the terms in the chain $M \supset \alpha M \supset \cdots \supset \alpha^j M \supset \cdots$ are all distinct and that the chain factors are all isomorphic. By the very definition, we must have $\mathrm{K}\dim M \geqq \mathrm{K}\dim M/\alpha M + 1$.

Assume now that the theorem holds for all r-sequences of length less than n, where $n > 1$, and that $\alpha_1, \cdots, \alpha_n$ is an r-sequence on M. From 2.1 it is clear that $\alpha_1, \cdots, \alpha_{n-1}$ is also an r-sequence on M. If N denotes the factor $M/\Sigma_{i=1}^{n-1} \alpha_i M$, then by induction $\operatorname{K dim} M \geqq \operatorname{K dim} N + (n-1)$. As discussed in 2.3(i), α_n induces a proper monomorphism of N, and so by the first part of the proof $\operatorname{K dim} N \geqq \operatorname{K dim}(N/\bar{\alpha}_n N) + 1$. As $N/\bar{\alpha}_n N \simeq \bar{M}$, combining these two inequalities gives the desired result.

As an application of 2.6, we see that the Weyl algebra A_n can contain no r-sequences of length greater than n. For $\operatorname{K dim} A_n = n$ as proven in [8, p. 74]. In the same vein, the theorem shows that the length of any r-sequence on M is bounded by $\operatorname{K dim} M$. This generalizes a well-known result for R-sequences [15, p.100]. There are, however, two important properties of R-sequences which do not hold for r-sequences.

EXAMPLE 2.7. (i) Let $U \subset R$ be a right ideal, $u_1, \cdots, u_n$ an r-sequence which is contained in U. Call this sequence *maximal in* U if there exists no $v \in U$ such that $u_1, \cdots, u_n, v$ is an r-sequence in R. When R is a commutative Noetherian ring, all maximal R-sequences in U have the same length [15, p. 97]. For r-sequences in noncommutative Noetherian rings, this fails.

Let k be a field of characteristic zero, $K = k[y]$ the ring of polynomials in a commuting indeterminate over k. Let σ be the k-automorphism $y \to y+1$ and $T = K[t;\sigma]$ the twisted polynomial ring (see [6, 12.2] for the definition and basic properties of these rings). Easily, t, y is a maximal r-sequence in $U = tT + yT$. If $u = t - y$, we claim that the sequence consisting of u alone is also maximal. For if u, v is an r-sequence then necessarily $vu \in uT$. By the right division algorithm for skew polynomial rings [10, p. 29], we can write $v = (t-y)q + r$ where $q \in T$ and $r \in K$. A relation $vu = uh$ then implies $r(t-y) = (t-y)s$ and a comparison of degrees shows that $s \in K$. Equating coefficients we get $s = \sigma(r)$, $ry = ys$, whence $r = s \in K^\sigma = k$. It follows that $\operatorname{End}_T(T/uT) \simeq k$ (see 3.9), and that T/uT admits no proper monomorphisms.

(ii) Let R be a ring, U a right ideal of R. A generating set $\{u_1, \cdots, u_j\}$ for U is called *minimal* if every other system of generators has at least j elements. For commutative rings, the length of any R-sequence contained in U is bounded by the cardinality of a minimal generating set [13, p. 91]. Stafford [21] has shown that in the Weyl algebra A_n any right ideal containing the element p_n can be generated by p_n and one other element. In particular, the right ideal $p_1 A_n + \cdots + p_n A_n$ contains an r-sequence of length n and yet can be generated by two elements.

Call an r-sequence commuting if $[\alpha_i, \alpha_j] = 0$ for all i, j. We let $\mathrm{pd}(M)$ denote the projective or homological dimension of an R-module M and refer the reader to [13, part III] for the definition. The remainder of this section is devoted to proving the following theorem.

THEOREM 2.8. *Let $u_1, \cdots, u_n$ be a commuting r-sequence in the ring R and let $U = u_1R + \cdots + u_nR$. Then* $\mathrm{pd}(R/U) = n$ *iff $Ru_1 + \cdots + Ru_n \neq R$.*

The proof requires a preparatory lemma and depends on an important tool of homological algebra — the Koszul complex. The construction outlined here is taken from Northcott [16, chapter 8].

The basic ingredients of the Koszul complex (more precisely, the Koszul complex on M with respect to $\alpha_1, \cdots, \alpha_n$) are a right R-module M and a commuting set of endomorphisms of M. For each integer j, $0 \leqq j \leqq n$, let I_j denote the set of j-tuples of integers $(\nu) = (\nu_1, \cdots, \nu_j)$ satisfying $1 \leqq \nu_1 < \nu_2 < \cdots < \nu_j \leqq n$. There are precisely $\binom{n}{j}$ elements in the set I_j. Let $T_1, \cdots, T_n$ be n "indeterminates" and to each $(\nu) \in I_j$ associate the monomial $T_{(\nu)} = T_{\nu_1}T_{\nu_2}\cdots T_{\nu_j}$. Let

$$X_j = \left\{ \sum_{(\nu)} T_{(\nu)}m_{(\nu)} \,\middle|\, (\nu) \in I_j,\ m_{(\nu)} \in M \right\},$$

that is, X_j is the set of all formal linear combinations of the $T_{(\nu)}$ with coefficients in M. X_j is a right R-module in the obvious way. Indeed, X_j is isomorphic to the direct sum $M^{\binom{n}{j}}$ and each $x \in X_j$ has a unique expression $x = \Sigma_{(\nu)} T_{(\nu)}m_{(\nu)}$. For $1 \leqq j \leqq n$ define maps $d_j : X_j \to X_{j-1}$ (here $X_0 = M$; below $\hat{T}_{\nu_p}$ denotes deletion of T_{ν_p}) by letting

$$d_j(T_{(\nu)}m) = d_j(T_{\nu_1}\cdots T_{\nu_j}m) = \sum_{p=1}^{j} (-1)^{p-1} T_{\nu_1}\cdots \hat{T}_{\nu_p} \cdots T_{\nu_j}(\alpha_{\nu_p}m)$$

for each monomial $T_{(\nu)}m$ and then extending by linearity. As the α_i are R-linear, so too is each d_j and one thus obtains a sequence of R-modules. More importantly, one has the following result.

LEMMA 2.9. *Let M be an R-module, $\{\alpha_1, \cdots, \alpha_n\}$ a commuting set of endomorphisms of M. Let X_j and d_j be as in the preceding paragraph. Then*

(i) $0 \to X_n \xrightarrow{d_n} \cdots \to X_1 \xrightarrow{d_1} X_0$ *is a complex of R-modules;*

(ii) *if $\alpha_1, \cdots, \alpha_n$ is a commuting r-sequence on M, the complex in* (i) *is acyclic and the following sequence is exact:*

$$0 \to X_n \xrightarrow{d_n} \cdots \to X_1 \xrightarrow{d_1} X_0 \to M \Big/ \sum_{i=1}^{n} \alpha_i M \to 0.$$

PROOF. (i) One need only replace the phrase "central elements of R" by "commuting endomorphisms of M" to use the argument given in [16, pp. 358–360]. Commutativity of the α_i is essential here, however.

(ii) Northcott's methods apply, but not without modification. Let $\mathbf{X}$ denote the Koszul complex on M with respect to $\{\alpha_1, \cdots, \alpha_n\}$, $\mathbf{Y}$ that with respect to $\{1, \alpha_2, \cdots, \alpha_n\}$. By [16, lemma 2, p. 363] the identity map admits a contracting homotopy and $\mathbf{Y}$ is acyclic. Now let $\bar{\mathbf{X}}$ denote the Koszul complex on $M/\alpha_1 M$ with respect to the induced maps $\{\bar{\alpha}_2, \cdots, \bar{\alpha}_n\}$, and let $\mathbf{Z}$ be the complex $\bar{\mathbf{X}}$ "displaced one place to the left" [16, p. 367]. There are chain maps f, g such that the sequence of complexes

$$0 \to \mathbf{X} \xrightarrow{f} \mathbf{Y} \xrightarrow{g} \mathbf{Z} \to 0$$

is exact [16, pp. 366–368]. Taking homology, one gets the usual long exact sequence

$$(*) \qquad \cdots \to H_{j+1}(\mathbf{Y}) \to H_{j+1}(\mathbf{Z}) \to H_j(\mathbf{X}) \to H_j(\mathbf{Y}) \to \cdots.$$

As $\mathbf{Y}$ is acyclic, $H_j(\mathbf{Y}) = 0$ for all j. As $Z_{j+1} \approx \bar{X}_j$, $H_{j+1}(\mathbf{Z}) \approx H_j(\bar{\mathbf{X}})$. Making these substitutions in $(*)$ gives

$$0 \to H_j(\bar{\mathbf{X}}) \to H_j(\mathbf{X}) \to 0$$

whence $H_j(\bar{\mathbf{X}}) \approx H_j(\mathbf{X})$ for all j. The induction argument of [16, theorem 7, p. 373] now shows that $H_j(\mathbf{X}) = 0$ for all $j \neq 0$. Thus when $\alpha_1, \cdots, \alpha_n$ is a commuting r-sequence, the complex in (i) is acyclic. From the definition above, $\operatorname{Im} d_1 = \alpha_1 M + \cdots + \alpha_n M$ and the sequence in (ii) is also exact.

For a terser proof using exterior algebras, see [5, pp. 151–152].

PROOF OF THEOREM 2.8. ($\Leftarrow$) From 2.9, the Koszul complex $\mathbf{X}$ is a deleted projective resolution of R/U. As each X_j is free, one has the usual identification $\operatorname{Hom}(X_j, R) \stackrel{\varepsilon}{=} R^{\binom{n}{j}}$. In detail, if $\{e_i\}$ is the canonical basis for the free right R-module R^k, and $\phi \in \operatorname{Hom}(R^k, R)$ then $\varepsilon(\phi) = (\phi e_1, \cdots, \phi e_k)$. Applying $\operatorname{Hom}(\ \ , R)$ to $\mathbf{X}$ and letting $\rho_j = \varepsilon d_j^* \varepsilon^{-1}$, one obtains an equivalence of complexes:

$$\begin{array}{ccccccccc} 0 \to & \operatorname{Hom}(X_0, R) & \xrightarrow{d_1^*} \cdots \to & \operatorname{Hom}(X_{n-1}, R) & \xrightarrow{d_n^*} & \operatorname{Hom}(X_n, R) & \to 0 \\ & \downarrow \varepsilon & & \downarrow \varepsilon & & \downarrow \varepsilon & \\ 0 \to & R & \xrightarrow{\rho_1} \cdots \to & R^n & \xrightarrow{\rho_n} & R & \to 0. \end{array}$$

In particular, as $X_n \xrightarrow{d_n} X_{n-1}$ is defined (with the obvious change in notation) by

$$d_n(a) = (u_1 a, -u_2 a, \cdots, (-1)^{n-1} u_n a)$$

the map ρ_n is given by $\rho_n(a_1, \cdots, a_n) = \Sigma_{i=1}^n (-1)^{i-1} a_i u_i$. Applying the definition of Ext_R^n and translating via ε, we get

$$\mathrm{Ext}_R^n(R/U, R) = \frac{\mathrm{Ker}\, d_{n+1}^*}{\mathrm{Im}\, d_n^*} \approx \frac{\mathrm{Ker}\, \rho_{n+1}}{\mathrm{Im}\, \rho_n} = \frac{R}{Ru_1 + \cdots + Ru_n}.$$

By assumption this is nonzero and $\mathrm{pd}(R/U) \geqq n$. The reverse ineqûality follows from 2.9, which shows that R/U admits a free resolution of length n.

($\Rightarrow$) We show that if $Ru_1 + \cdots + Ru_n = R$, then $\mathrm{pd}(R/U) \leqq n - 1$. By 2.9 there is an exact sequence

$$(*) \qquad 0 \to \mathrm{Im}\, d_n \to R^n \to \mathrm{Im}\, d_{n-1} \to 0.$$

If $\tilde{u}$ denotes the vector $\tilde{u} = (u_1, -u_2, \cdots, (-1)^{n-1} u_n)$, then from the description of d_n given above $\mathrm{Im}\, d_n = \tilde{u}R$. By assumption there exist $b_i \in R$ such that $b_1 u_1 + \cdots + b_n u_n = 1$. Let $\{e_1, \cdots, e_n\}$ denote the usual basis for R^n and define $q : R^n \to \mathrm{Im}\, d_n$ by $q(e_i) = (-1)^{i-1} \tilde{u} b_i$. Then if $j : \mathrm{Im}\, d_n \to R^n$ is the inclusion,

$$qj(\tilde{u}) = q\left(\sum_{i=1}^n (-1)^{i-1} e_i u_i\right) = \sum_{i=1}^n (-1)^{i-1}(-1)^{i-1} \tilde{u} b_i u_i = \tilde{u}\left(\sum_{i=1}^n b_i u_i\right) = \tilde{u}.$$

Thus $(*)$ splits and $\mathrm{Im}\, d_{n-1} = \mathrm{Ker}\, d_{n-2}$ is projective.

EXAMPLE 2.10. Let k be a field, V a k-space with countable basis $\{v_i\}_{i=1}^\infty$, and let $R = \mathrm{End}_k(V)$. Let $a \in R$ be the shift transformation $a(v_i) = v_{i+1}$, and let $b \in R$ be defined by $b(v_1) = 0$, $b(v_i) = v_{i-1}$ for $i > 1$. Then $ba = 1$ but $ab \neq 1$. The first equation implies that a is right regular, the second that $R = aR \oplus (1 - ab)R$. Thus, although a is an r-sequence of length 1, R/aR is projective. Similarly in $T = R[x]$, the elements x, a constitute an r-sequence of length 2 but $\mathrm{pd}(T/xT + aT) = 1$.

These examples, which were pointed out to me by W. Blair, show that the hypothesis $\Sigma_i Ru_i \neq R$ in 2.8 is nonvacuous. They also provide a counterexample to Theorem 4.2 in Cartan–Eilenberg [5, p. 150]. The statement of 2.8 given there omits the above hypothesis, and the claimed isomorphism $\mathrm{Ext}_R^n(R/U, R/U) \approx R/U$ [5, p. 153] does not always hold.

REMARK 2.11. Let $u_1, \cdots, u_n$ be a not necessarily commuting r-sequence in R, $U = u_1 R + \cdots + u_n R$. From standard results in homological algebra [13, p. 169], $\mathrm{pd}(R/U) \leqq n$. Walker [23, p. 28] has shown that equality also holds if $u_1, \cdots, u_n$ is a "regular normalizing set" rather than a commuting r-sequence. His argument uses the fact that a normalizing element generates a two-sided ideal in R and is independent of our Theorem 2.8; nonetheless, the condition $\Sigma_i Ru_i \neq R$ is manifestly satisfied. It is interesting to speculate if this condition alone implies $\mathrm{pd}(R/U) = n$.

3. Transcendental algebras and $D(x_1, \cdots, x_n)$

In this section we prove the theorem stated in the introduction. It appears here as Theorem 3.16.

The section begins with the definition of the transcendence degree of an algebra (3.1). We then show that if A is transcendental, $A(x_1, \cdots, x_n) = A \otimes_k k(x_1, \cdots, x_n)$ is never Artinian (3.4). If D is a division ring which is transcendental over its center, then the preceding result and a classical theorem on central simple algebras (3.5) give at once an infinite family of nonisomorphic, nonartinian, simple Noetherian domains. The question as to the Krull and global dimensions of these rings arises naturally. While the inequality $1 \leqq \mathrm{K}\dim[D(x_1, \cdots, x_n)] \leqq n$ is easily established (3.8), the dimension can in fact be any integer in the set $\{1, 2, \cdots, n\}$ (see 4.2). It was precisely this problem which motivated Definition 3.1. After some technical preliminaries on polynomial rings and their r-sequences, we show that if the transcendence degree of D is sufficiently large the ring $D(x_1, \cdots, x_n)$ has dimension n.

Throughout this section k is a field, A a k-algebra and R an arbitrary ring. No restrictions are imposed on the characteristic of k and all tensor products are relative to k.

Definition 3.1. (i) Let A be a k-algebra. A is said to be *transcendental* over k if A is not an algebraic k-algebra. Clearly, A is transcendental over k if and only if the "structure map" $k \to A$ extends to an embedding of the polynomial ring $k[x]$ into A.

(ii) The transcendence degree of A over k, denoted tr. deg. (A/k), is defined by

$$\text{tr. deg.}(A/k) = \sup\{\text{tr. deg.}(C/k) \mid C \supset k \text{ a commutative subdomain of } A\}.$$

Here, tr. deg. (C/k) is defined as the transcendence degree of L over k, where L is the field of fractions of C.

Definition 3.2. Let A, k be as above and let $k(x_1, \cdots, x_n)$ be the rational function field in n variables over k. $A(x_1, \cdots, x_n)$ denotes the tensor product $A \otimes k(x_1, \cdots, x_n)$.

$A(x_1, \cdots, x_n)$ is isomorphic to the localization of the polynomial ring

$A[x_1, \cdots, x_n]$ at the central multiplicative set $S = k[x_1, \cdots, x_n] - \{0\}$. When A is prime Goldie, say, it can be viewed as a subring of Q, the classical ring of quotients of $A[x_1, \cdots, x_n]$. In general it is a proper subring of Q. Our use of the notation $A(x_1, \cdots, x_n)$ is thus a little nonstandard; hopefully, this will not cause undue confusion.

An important case where $A(x_1, \cdots, x_n)$ is the full ring of quotients of $A[x_1, \cdots, x_n]$ is when A is finite dimensional over k. If "finite dimensional" is weakened to "algebraic" then it is unknown in general whether $A(x_1, \cdots, x_n)$ is Artinian [11, pp. 240–241]. When A is transcendental over k it will be shown presently that $A(x_1, \cdots, x_n)$ is never Artinian. Before proceeding to the proof, we need to recall a few facts about polynomial rings and to clarify some notation. In $R[x]$ one has the freedom of writing ceofficients to the left or to the right of the indeterminate. Since all our modules are right R-modules, we choose the right normalization.

LEMMA 3.3. *Let R be a ring, $R[x]$ the ring of polynomials over R, and $t \in R$.*

(i) (Right Remainder Theorem) *Given $f \in R[x]$, there exist unique $r \in R$, $q \in R[x]$ such that $f = r + (x-t)q$. Moreover, if $f = \Sigma_i x^i a_i$ then $r = \Sigma\, t^i a_i = f(t)$.*

(ii) *If $\varepsilon : R[x] \to R$ is defined by $\varepsilon(\Sigma_i x^i a_i) = \Sigma_i t^i a_i$, then ε is a map of R-modules and* $\operatorname{Ker}\varepsilon = (x-t)R[x]$.

(iii) *If R is a faithful algebra over the commutative domain K, then t is transcendental over K iff $(x-t)R[x] \cap K[x] = 0$.*

(iv) *If R, K are as in* (iii) *and $t \in R$ is transcendental over K, then t is transcendental over $K[x_1, \cdots, x_n]$ in $R[x_1, \cdots, x_n]$.*

PROOF. (i) This is a special case of the (right) division algorithm which is valid for any ring as long as the divisor is monic. The last assertion follows from the identity

$$x^j - t^j = (x-t)(x^{j-1} + x^{j-2}t + \cdots + t^{j-1}).$$

(ii) Since $R[x]$ is a free right R-module with basis $\{x^i \mid i \geqq 0\}$, ε is well-defined and R-linear. That $\operatorname{Ker}\varepsilon = (x-t)R[x]$ follows from (i).

(iii) Just as for fields, if the ring R contains a commutative domain K in its center, we say that $t \in R$ is transcendental over K if and only if $0 \neq f \in K[x] \Rightarrow f(t) \neq 0$.

Take ε as in (ii) and restrict to $K[x]$. Whereas ε is not a ring homomorphism, the restriction is and its kernel is $\operatorname{Ker}\varepsilon \cap K[x]$.

(iv) This can be proven by induction on the number of variables and is left to the reader.

LEMMA 3.4. *Let A be a k-algebra. If A is trancendental over k then $A(x_1, \cdots, x_n)$ is never Artinian.*

PROOF. Let $t \in A$ be transcendental. By 3.3 $(x_n - t)A[x_1, \cdots, x_n] \cap k[x_1, \cdots, x_n] = 0$, and so $x_n - t$ generates a nonzero proper right ideal in $A(x_1, \cdots, x_n)$. The element $x_n - t$ is thus regular but not invertible; such elements do not exist in Artinian rings.

The following result is quite well-known but is included for the sake of completeness.

PROPOSITION 3.5. *Let A be a central simple k-algebra.*

(i) *If B is an arbitrary k-algebra, the map $I \to A \otimes I$ defines a one-to-one correspondence between the set of two-sided ideals of B and that of $A \otimes B$.*

(ii) *If F is an extension field of k, then $A_F = A \otimes F$ is a central simple F-algebra.*

PROOF. (i) See [24, p. 98], [6, p. 363].

(ii) We have simplicity of A_F by (i). That $\mathrm{Cen}(A_F) = F$ is a special case of the isomorphism $\mathrm{Cen}(G \otimes H) \simeq \mathrm{Cen}(G) \otimes \mathrm{Cen}(H)$ [6, p. 62], valid for any k-algebras G, H.

Theorem 3.6 is an easy consequence of 3.4 and 3.5, but was considered important enough to warrant explicit statement. It gives a substantial broadening of a class of simple Noetherian rings described in [7, pp. 61–63]. The hypothesis that A be transcendental is not as restrictive as one might expect — for a simple Noetherian ring to be transcendental over its center it suffices by Goldie's Theorem that it have nonzero Krull dimension. The interesting and open question is whether the converse is true: If $A(x_1, \cdots, x_n)$ is Noetherian but not Artinian, is A necessarily transcendental? This is true, for example, if k is uncountable [1, p. 47].

THEOREM 3.6. *Let A be a central simple Noetherian k-algebra. If A is transcendental over k then $A(x_1, \cdots, x_n)$ is simple Noetherian, not Artinian, for every integer n.*

We turn now to the Krull and global dimension of $A(x_1, \cdots, x_n)$. Examples show that the bounds given in 3.8 are, in general, best possible.

LEMMA 3.7. *For each integer $j \leqq n$, $A(x_1, \cdots, x_n)$ is a free (left or right) $A(x_1, \cdots, x_j)$-module.*

PROOF. Fix n and j and put $B = A(x_1, \cdots, x_j)$, $F = k(x_1, \cdots, x_j)$, $E = k(x_1, \cdots, x_n)$. As F is a field, E is a free F-module and so isomorphic to $\bigoplus_I F$ for some index set I. Standard properties of tensor products give the following chain of isomorphisms:

$$A \bigotimes_k E \simeq A \bigotimes_k \left(E \bigotimes_F F\right) \simeq \left(A \bigotimes_k F\right) \bigotimes_F E \simeq B \bigotimes_F \left(\bigoplus_I F\right) \simeq \bigoplus_I B.$$

An examination of the above proof shows that one can choose an A-basis for $A(x_1, \cdots, x_n)$ which consists of elements of $k(x_1, \cdots, x_n)$ and contains 1. This has two useful consequences: (1) $UA(x_1, \cdots, x_n) \cap A = U$ for any right ideal $U \subset A$, (2) A is an A-bimodule summand of $A(x_1, \cdots, x_n)$.

PROPOSITION 3.8. *Let A be a k-algebra.*

(i) *If A is Noetherian,* $\mathrm{K}\dim A \leqq \mathrm{K}\dim[A(x_1, \cdots, x_n)] \leqq \mathrm{K}\dim A + n$.

(ii) $\mathrm{gl.\,dim}(A) \leqq \mathrm{gl.\,dim}[A(x_1, \cdots, x_n)] \leqq \mathrm{gl.\,dim}(A) + n$.

PROOF. (i) By the preceding observation, the map $U \to UA(x_1, \cdots, x_n)$ defines an embedding of the set of right ideals of A into that of $A(x_1, \cdots, x_n)$. The first inequality is thus a consequence of [8, (a), p. 712]. As is well-known, $\mathrm{K}\dim(A[x_1, \cdots, x_n]) = \mathrm{K}\dim A + n$ [8, p. 714]. Since $A(x_1, \cdots, x_n)$ is a localization of this ring, $V \to V \cap A[x_1, \cdots, x_n]$ is an embedding of the lattice of right ideals of $A(x_1, \cdots, x_n)$ into that of $A[x_1, \cdots, x_n]$. The result of Gabriel–Rentschler cited above gives the second inequality.

(ii) The first inequality follows from [13, theorem 5, p. 173]. For the second, note that $\mathrm{gl.\,dim}(A[x_1, \cdots, x_n]) = \mathrm{gl.\,dim}\, A + n$ [13, p. 174] and that the dimension at worst goes down upon localization [13, p. 181].

Independently of 3.8 we will show below that if tr. deg.$(A/k) \geqq n$ then the dimensions of $A(x_1, \cdots, x_n)$ are at least n. The proof depends on the results of Section 2 and the following observation: if $\{t_1, \cdots, t_n\}$ is a commuting, algebraically independent subset of A then $x_1 - t_1, \cdots, x_n - t_n$ is a commuting r-sequence in $A(x_1, \cdots, x_n)$.

DEFINITION 3.9. Let R be a ring, $V \subset R$ a right ideal. The *idealizer of V*, denoted $\mathbf{I}_R(V)$, is defined by

$$\mathbf{I}_R(V) = \{a \mid aV \subset V\}.$$

By an argument which dates back to Fitting, the multiplication map $R \to \mathrm{End}_R(R)$ induces an isomorphism $\mathbf{I}(V)/V \simeq \mathrm{End}_R(R/V)$ [18, p. 309].

Now suppose that $C \subset R$ is an *additive* subgroup. Let $B = \{a \mid aC \subset C\}$. Then B is a subring of R, the "stabilizer" of C, and abusing the above notation we write $B = \mathbf{I}(C)$. Note that if $C[x] = \{\Sigma_i x^i c_i \mid c_i \in C\}$, then $C[x]$ is a subgroup of $R[x]$ and $\mathbf{I}(C[x]) = B[x]$. We take C to be an additive sugroup in Lemma

3.10 since it combines the two cases of interest: C a right ideal of R or C a subring of R.

LEMMA 3.10. *Let R, C, B be as above. Let g be a monic polynomial in $B[x]$ and let $H = g^{-1}(C[x]) = \{h \mid gh \in C[x]\}$. Then $H = C[x]$.*

PROOF. The proof is by induction on the degree of h, where $h \in H$. If $\deg h = 0$ and $gh = c$ for some $c \in C[x]$, then a comparison of leading coefficients shows that $h \in C$. Assume that the result holds for polynomials of degree less than n. Let $h \in H$ have degree n, and write $h = h' + x^n a_n$. Taking a relation $gh = c$ and again comparing leading coefficients, we get $a_n \in C$. Now $gx^n a_n \in C[x]$ since $g \in B[x]$, and $gh' = gh - gx^n a_n \in C[x]$. By induction $h' \in C[x]$, and so too is $h = h' + x^n a_n$.

LEMMA 3.11. *Let $u_1, \cdots, u_n$ be an r-sequence in R. Put $U = \sum_{i=1}^n u_i R$ and let $B = \mathbf{I}(U)$.*

(i) *$u_1, \cdots, u_n$ is an r-sequence in $R[x]$.*

(ii) *If g is a monic, nonconstant polynomial in $B[x]$, then $u_1, \cdots, u_n$, g is an r-sequence in $R[x]$.*

PROOF. (i) Verifying the conditions of 2.1 is a matter of applying the appropriate definitions, and is left to the reader.

(ii) In view of (i) it suffices to show that:

(1) $V = UR[x] + gR[x] \neq R[x]$,

(2) $g(UR[x]) \subset UR[x]$, and

(3) $H = \{h \mid gh \in UR[x]\} = UR[x]$.

To prove (1), suppose that $gf + u = 1$ where $f \in R[x]$, $u \in UR[x]$. Then $gf = 1 - u \in B[x]$. From 3.10 with $C = B$, whence $B = \mathbf{I}(B)$, we see that f is in fact an element of $B[x]$. Reducing modulo $UB[x] = UR[x]$ then gives $\overline{gf} = \bar{1}$; by our assumption on g this is absurd. For (2) note that $\mathbf{I}(UR[x]) = B[x]$, and for (3) use 3.10 with $C = U$.

LEMMA 3.12. *Let A be a k-algebra, $\{t_1, \cdots, t_n\}$ a commuting subset of A. In $A[x_1, \cdots, x_n]$ let $u_i = x_i - t_i$ and $U = \sum_i u_i A[x_1, \cdots, x_n]$. Then $U \cap k[x_1, \cdots, x_n] = 0$ if and only if the set $\{t_1, \cdots, t_n\}$ is algebraically independent over k.*

PROOF. Let $\phi : A[x_1, \cdots, x_n] \to A$ be the A-module map defined by $x_1^{\nu_1} \cdots x_n^{\nu_n} \to t_1^{\nu_1} \cdots t_n^{\nu_n}$. By restriction, ϕ induces a ring homomorphism of $k[x_1, \cdots, x_n]$ onto $k[t_1, \cdots, t_n]$; our theorem is proven if we show $\operatorname{Ker} \phi = U$.

Proving this equality is complicated by the inability to apply the division algorithm directly. To simplify matters, consider first the following situation: $V \subset A$ is a right ideal, $t \in \mathbf{I}(V)$, and $U = (x - t)A[x] + VA[x]$. Let $\varepsilon : A[x] \to A$ be the evaluation map defined in 3.3(ii). We claim that $\varepsilon^{-1}(V) = U$. Suppose first that $f \in U$. Then $f = (x - t)h + \Sigma x^i v_i$ where $v_i \in V$; since $\operatorname{Ker} \varepsilon = (x - t)A[x]$ and $t \in \mathbf{I}(V)$, $\varepsilon f = \Sigma t^i v_i \in V$. Conversely, suppose $\varepsilon f \in V$. By 3.3, $f = (x - t)q + \varepsilon f$, and this is manifestly in U. In the general case, for $1 \leqq j \leqq n$, let $A_j = A[x_1, \cdots, x_j]$, $U_j = \Sigma_{i=1}^j u_i A_j$ and define $\varepsilon_j : A_j \to A_{j-1}$ by $\varepsilon_j(x_j) = t_j$. With this notation, $U_j = (x_j - t_j)A_{j-1}[x_j] + U_{j-1}A_{j-1}[x_j]$ and $t_j \in \mathbf{I}_{A_{j-1}}(U_{j-1})$. The preceding argument shows that $\varepsilon_j^{-1}(U_{j-1}) = U_j$. As $\phi = \varepsilon_1 \varepsilon_2 \cdots \varepsilon_n$, $\phi^{-1}(0) = U_n = U$.

As a corollary to 3.12 we obtain a sufficient condition for the primitivity of a polynomial ring over a division algebra. The result has since been generalized by Amitsur and Small, who have determined necessary and sufficient conditions for primitivity of $D[x_1, \cdots, x_n]$ [4].

THEOREM 3.13. *Let D be a division ring with center k. If* $\operatorname{tr.deg.}(D/k) \geqq n$, *then $D[x_1, \cdots, x_n]$ is primitive.*

PROOF. Let $t_1, \cdots, t_n$ be n commuting transcendentals in D and let u_i, U, and ϕ be as above. By the proof of 3.12 $\operatorname{Ker} \phi = U$, and clearly $\operatorname{Im} \phi = D$. U is thus a maximal D-submodule of $D[x_1, \cdots, x_n]$, all the more a maximal $D[x_1, \cdots, x_n]$-submodule.

Letting $A = D$, $B = k[x_1, \cdots, x_n]$ in 3.5, we see that all the two-sided ideals of $D[x_1, \cdots, x_n]$ are extended ideals of $k[x_1, \cdots, x_n]$. By 3.12, $U \cap k[x_1, \cdots, x_n] = 0$, hence $D[x_1, \cdots, x_n]/U$ is faithful.

LEMMA 3.14. *Let A, t_i, u_i and U be as in Lemma 3.12. Then*

(i) *$u_1, \cdots, u_n$ is a commuting r-sequence in $A[x_1, \cdots, x_n]$.*

(ii) *$u_1, \cdots, u_n$ is an r-sequence in $A(x_1, \cdots, x_n)$ if and only if $\{t_1, \cdots, t_n\}$ is algebraically independent over k.*

PROOF. (i) That $u_1, \cdots, u_n$ is an r-sequence follows by Lemma 3.11 and induction on the number of variables. Since $[u_i, u_j] = [t_i, t_j]$, commutativity is obvious.

(ii) By 2.5, $u_1, \cdots, u_n$ is an r-sequence in $A(x_1, \cdots, x_n)$ if and only if $UA(x_1, \cdots, x_n)$ is proper; by 3.12 this holds if and only if $\{t_1, \cdots, t_n\}$ is algebraically independent over k.

THEOREM 3.15. *Let A be a k-algebra with* $\operatorname{tr.deg.}(A/k) \geqq n$.

(i) *If A is Noetherian,* $\operatorname{K dim}[A(x_1, \cdots, x_n)] \geqq n$.

(ii) $\operatorname{gl.dim}[A(x_1, \cdots, x_n)] \geqq n$.

PROOF. By the lemma, $A(x_1, \cdots, x_n)$ contains a commuting r-sequence of length n. The first assertion follows from 2.6, the second from 2.9.

Before stating Theorem 3.16, we note one application of the more general 3.15. Let X be any set and let $k\langle X\rangle$ denote the free associative k-algebra on X. Since $\mathrm{gl.\,dim}(F\langle X\rangle) = 1$ for any field F [6, pp. 144–145], Theorem 3.15 and the isomorphism $k\langle X\rangle \otimes k(x_1, \cdots, x_n) \cong [k(x_1, \cdots, x_n)]\langle X\rangle$ imply that any commutative subalgebra of $k\langle X\rangle$ has transcendence degree at most one over k. This gives an alternate proof of a corollary to a theorem of Bergman [25].

THEOREM 3.16. *Let D be a division ring with center k. If* $\mathrm{tr.\,deg.}(D/k) \geqq n$, *then $D(x_1, \cdots, x_n)$ is a simple Noetherian domain of Krull and global dimensions n.*

PROOF. Combine 3.6, 3.8, and 3.15.

REMARK 3.17. (i) Theorem 3.16 is actually a special case of a slightly more general theorem: If A is (semisimple) Artinian and $\mathrm{tr.\,deg.}(A/k) \geqq n$, then $(\mathrm{gl.\,dim})\,\mathrm{K\,dim}[A(x_1, \cdots, x_n)] = n$.

(ii) The conclusions of 3.16 remain true if the hypothesis "$\mathrm{tr.\,deg.}(D/k) \geqq n$" is replaced by "$D[x_1, \cdots, x_n]$ primitive". For by recent work of Amitsur and Small [4], if $D[x_1, \cdots, x_n]$ is primitive then $k[x_1, \cdots, x_n]$ embeds in $M_l(D)$, the ring of $l \times l$ matrices over D, for some (undetermined) integer l. From (i) it follows that the algebra $[M_l(D)](x_1, \cdots, x_n)$ has both dimensions n, and since these are Morita invariants so does $D(x_1, \cdots, x_n)$.

4. Computing the transcendence degree of a division ring

Comparing Theorem 3.16 with 3.6, it is natural to ask if the full strength of the hypothesis $\mathrm{tr.\,deg.}(D/k) \geqq n$ is actually necessary in the former. It was the attempt to answer this question which led to the results of this section. The question itself is easily and decisively settled by Theorem 4.2, which shows that the mere presence of transcendentals in D is insufficient for the conclusions of Theorem 3.16. The same example also gave the first hint of a more interesting and unexpected phenomenon — Theorem 3.16 can often be used to explicitly compute the transcendence degree of a division ring. In this section we will determine $\mathrm{tr.\,deg.}(D)$ for the quotient division rings of three important families of noncommutative domains: (1) the Weyl algebras (4.2); (2) twisted Laurent polynomial rings (4.3); and (3) rings of differential operators (4.8).

As in the preceding section, k is a field, A a k-algebra and $A(x_1, \cdots, x_n) = A \otimes k(x_1, \cdots, x_n)$. Appropriate restrictions upon the characteristic of k will be noted as needed.

If we are to compute the transcendence degree of a division ring, we must necessarily be able to identify its center. The following proposition, which is part of the folklore, gives an important case where such identification is always possible.

PROPOSITION 4.1. *Let R be a simple ring, $S \subset R$ a right Ore set, R_S the ring of fractions. Then* $\mathrm{Cen}(R_S) = \mathrm{Cen}(R)$.

PROOF. Let $q \in \mathrm{Cen}(R_S)$ and let $I = \{r \in R \mid qr \in R\}$. Since q is central, I is an ideal of R. By the definition of R_S, $q = as^{-1}$ where $a \in R$, $s \in S$, and so $I \neq 0$. As R is simple, $I = R$ and $q \in qR \subset R$. This gives $\mathrm{Cen}(R_S) \subset \mathrm{Cen}(R)$ and the reverse inclusion is obvious.

Our first theorem shows that the Weyl algebra A_m and its quotient division ring D_m contain no commutative subdomains of transcendence degree greater than m. The result is known for A_1 and a more precise statement is possible; see [2, theorem 1, p. 2].

THEOREM 4.2. *Let k be a field of characteristic zero. Let A_m denote the mth Weyl algebra over k and D_m the quotient division ring of A_m.*

(i) *A_m is a central simple k-algebra.*

(ii) $\mathrm{K}\dim[A_m(x_1,\cdots,x_n)] = m = \mathrm{gl.}\dim[A_m(x_1,\cdots,x_n)]$.

(iii) $\mathrm{K}\dim[D_m(x_1,\cdots,x_n)] = \min\{n, m\} = \mathrm{gl.}\dim[D_m(x_1,\cdots,x_n)]$.

(iv) $\mathrm{tr.deg.}(D_m/k) = m = \mathrm{tr.deg.}(A_m/k)$.

PROOF. We recall again the basic definition: A_m is the k-algebra with generators $p_1,\cdots,p_m$; $q_1,\cdots,q_m$ and relations $[p_i,q_j] = \delta_{ij}$, $[p_i,p_j] = 0 = [q_i,q_j]$. This can be generalized to any ring R by defining $A_m(R)$ as follows. Let $R\langle x,y\rangle$ denote the free ring in two noncommuting variables over R and let I be the two-sided ideal generated by $f(x,y) = xy - yx - 1$. Then $A_1(R) = R\langle x,y\rangle/I$ and $A_m(R) = A_1(A_{m-1}(R))$. Standard properties of tensor products now show that $A_m(R) \simeq A_m(k)\otimes R$ for any k-algebra R. In particular, A_m can be identified with the m-fold tensor product $A_1\otimes\cdots\otimes A_1$.

(i) This result, of course, is quite well-known. For the standard argument that A_1 is central simple, see [6, p. 440]. To obtain central simplicity for $m > 1$, use the above description of A_m, Proposition 3.5, and induction.

(ii) From the remarks preceding (i),

$$A_m(x_1,\cdots,x_n) = A_m \otimes k(x_1,\cdots,x_n) \simeq A_m(k(x_1,\cdots,x_n)).$$

As proven in [8, p. 714], $\mathrm{K}\dim[A_m(E)] = m$ for any field E of characteristic zero. By [19], $\mathrm{gl.}\dim[A_m(E)] = m$ also.

(iii) First suppose $n \leqq m$. As $k[p_1, \cdots, p_m] \simeq k[x_1, \cdots, x_m]$ and A_m is central simple over k, tr. deg. $(D_m/k) \geqq m$ by 4.1. $\mathrm{K}\dim[D_m(x_1, \cdots, x_n)] = n = \mathrm{gl.}\dim[D_m(x_1, \cdots, x_n)]$ follows from 3.16. Now suppose $n > m$, and put $R = D_m(x_1, \cdots, x_m)$, $F = k(x_1, \cdots, x_m)$. Replacing A by R and k by F in 3.8, we see that $\mathrm{K}\dim[R \otimes_F F(x_{m+1}, \cdots, x_n)] \geqq \mathrm{K}\dim R = m$, and similarly for global dimension. Since $D_m(x_1, \cdots, x_n) \simeq R \otimes_F F(x_{m+1}, \cdots, x_n)$, both its dimensions are at least m. But as $D_m(x_1, \cdots, x_n)$ is a localization of $A_m(x_1, \cdots, x_n)$, they are at most m.

(iv) That tr. deg. $(D_m/k) \geqq m$ was established in the previous paragraph. If tr. deg. $(D_m/k) > m$, then by 3.16, $\mathrm{K}\dim[D_m(x_1, \cdots, x_{m+1})] = m + 1$, contradicting (iii). That $m \leqq$ tr. deg. $(A_m/k) \leqq$ tr. deg. (D_m/k) is clear from Definition 3.1.

We now state the analogue of 4.2 for the ring of twisted Laurent polynomials $T = L[t, t^{-1}; \sigma]$ and its division ring of quotients D.

THEOREM 4.3. *Let L be a field, σ an automorphism of L of infinite order, and k the fixed field of σ. Let $T = L[t, t^{-1}; \sigma]$ and let D be the quotient division ring. Assume* tr. deg. $(L/k) = m$, *where $m \geqq 1$ but is not necessarily finite.*

(i) *$T(x_1, \cdots, x_n)$ is a central simple $k(x_1, \cdots, x_n)$-algebra.*

(ii) $\mathrm{K}\dim[T(x_1, \cdots, x_n)] = \min\{n, m\} = \mathrm{gl.}\dim[T(x_1, \cdots, x_n)]$.

(iii) $\mathrm{K}\dim[D(x_1, \cdots, x_n)] = \min\{n, m\} = \mathrm{gl.}\dim[D(x_1, \cdots, x_n)]$.

(iv) *If $m < \infty$, then* tr. deg. $(D/k) = m =$ tr. deg. (T/k).

The proof of 4.3 is not as direct as that of 4.2. The basic complication is that $K = L \otimes k(x_1, \cdots, x_n)$ is not isomorphic to a rational function field over L. This necessitates Proposition 4.4 where the dimensions of a tensor product of two fields are computed. To determine the dimensions of $K[t, t^{-1}; \sigma]$ we need the extension of a theorem of Hart [9] which is given in 4.5. We remark that although the "dim" referred to in both these propositions is the classical Krull dimension of commutative algebra [15, p. 71], for the rings considered in 4.4 and 4.5 it coincides with the "K dim" [8, p. 713].

PROPOSITION 4.4. *Let k be a field, E an extension field of k.*

(i) $\dim[E \otimes k(x_1, \cdots, x_n)] = \min\{\text{tr. deg.}(E/k), n\}$.

(ii) $\mathrm{gl.}\dim[E \otimes k(x_1, \cdots, x_n)] = \min\{\text{tr. deg.}(E/k), n\}$.

PROOF. (i) This is a special case of a recent result of Sharp [20]. We note, however, that a direct proof can be based on Lemma 3.12 and some standard commutative algebra.

(ii) We can identify $E \otimes k(x_1, \cdots, x_n)$ with a localization of the polynomial ring $R = E[x_1, \cdots, x_n]$. As R is a regular Noetherian ring, so is the localization.

But for a regular ring, $\dim(R) = \mathrm{gl.\,dim}(R)$ [15, pp. 130–131]. Thus (ii) is a consequence of (i).

THEOREM 4.5. *Let K be a commutative Noetherian domain of finite, but nonzero, global dimension. Let σ be an automorphism of K such that $T = K[t, t^{-1}; \sigma]$ is a simple ring. Then*

(i) $\mathrm{K\,dim}(T) = \dim(K)$.

(ii) $\mathrm{gl.\,dim}(T) = \mathrm{gl.\,dim}(K)$.

PROOF. The above result was proven by Hart [9] for the ring of differential polynomials $R = K[y; d]$. His arguments depend on three facts: (1) for any nonzero prime $P \subset K$ and right ideal $J \supsetneq PR$, $J \cap K \supsetneq P$; (2) there exists a partial quotient ring R_S of R which is a hereditary Noetherian domain; and (3) R is a free K-module. Consider the ring T. To prove (1), first define the "length" of any $h = a_m t^m + \cdots + a_n t^n \in T$ (we have temporarily waived the requirement of right normalization, since σ is an automorphism this is permissible) by $l(h) = n - m$. Now choose $h \in J - PT$ of minimal length; it obviously suffices to show that $l(h) = 0$. Suppose not. Since multiplying on the right by t^{-m} preserves length, we may assume $h = a_0 + \cdots + a_l t^l$, where $a_i \notin P$ for all i. For any $p \in P$ $hp - ph \in J$ and has shorter length, whence $hp - ph \in PT$. In particular, $a_l\sigma^l(p) - a_l p \in P$ and $\sigma^l(p) \in P$. If we let $I = P\sigma(P)\cdots\sigma^{l-1}(P)$, then I is σ-invariant and generates a proper ideal in T. This contradicts thes simplicity of T. For (2) let $T_S = F[t, t^{-1}; \sigma]$, where F is the field of fractions of K and σ is the usual extension. (3) is clear. The rest of Hart's arguments, in particular Theorems 2.5, 2.6, and 3.2 [9, pp. 342–344], go through with only a change in notation.

PROOF OF THEOREM 4.3. (i) Since L is a field and σ has infinite order, $T = L[t, t^{-1}; \sigma]$ is simple [10, p. 38]. A straightforward computation shows that $\mathrm{Cen}(T) = k$ and so T is central simple k-algebra. Since $T(x_1, \cdots, x_n) = T \otimes k(x_1, \cdots, x_n)$, Proposition 3.5 does the rest.

(ii) Let $K = L \otimes k(x_1, \cdots, x_n)$ and extend σ to K by putting $\sigma(\lambda \otimes p) = \lambda^\sigma \otimes p$. There is a homomorphism $T(x_1, \cdots, x_n) \xrightarrow{\phi} K[t, t^{-1}; \sigma]$ given by $\phi(\Sigma_i a_i t^i \otimes p) = \Sigma_i (a_i \otimes p) t^i$. This is nonzero and by (i), $\mathrm{Ker}\,\phi = 0$. Surjectivity of ϕ is clear and so $T(x_1, \cdots, x_n) \simeq K[t, t^{-1}; \sigma]$. By 4.4, $\mathrm{K\,dim}(K) = \min\{n, m\} = \mathrm{gl.\,dim}(K)$, and by our hypothesis on L this is strictly positive. Thus 4.5 applies and gives $\mathrm{K\,dim}(K[t, t^{-1}; \sigma]) = \mathrm{K\,dim}(K)$, $\mathrm{gl.\,dim}(K[t, t^{-1}; \sigma]) = \mathrm{gl.\,dim}(K)$.

(iii) We saw in (i) that $\mathrm{Cen}(T) = k$. As T is simple, $\mathrm{Cen}(D) = k$ and $\mathrm{tr.\,deg.}(D/k) \geqq m$. If $n \leqq m$, then $\mathrm{K\,dim}[D(x_1, \cdots, x_n)] = n$ by 3.16. If $n > m$

then, just as in the proof of 4.2(iii), $m \leqq \mathrm{K}\dim[D(x_1,\cdots,x_n)] \leqq m$. The argument for global dimension is the same.

(iv) We have $\operatorname{tr.deg.}(D/k) \geqq m$. If $\operatorname{tr.deg.}(D/k)$ were strictly greater, then $\mathrm{K}\dim[D(x_1,\cdots,x_{m+1})] = m+1$ ($m < \infty$ here), contradicting (iii).

It is worth remarking that L is a maximal subfield of D. This can be seen by embedding D in the Laurent series ring $\Delta = L((t;\sigma))$ and showing that $C_\Delta(L) = L$.

We now show that for every integer $m \geqq 1$ there exists a field L satisfying all the hypotheses of 4.3. The construction which follows was suggested to me by A. Wadsworth.

EXAMPLE 4.6. Let $p_1,\cdots,p_m$ be m distinct primes in $\mathbf{Z}$ and let $\mathbf{Q}$ denote the field of rational numbers. Let $C = \mathbf{Q}[x_1,\cdots,x_m]$, L the field of fractions. Define $\sigma: C \to C$ by $x_i \to p_i x_i$ and extend σ to L as usual. We claim that $L^\sigma = \mathbf{Q}$.

First note that if $f \in C$ and $f^\sigma = uf$ where $u \in \mathbf{Q}$, then f has the form $f = ax_1^{\nu_1}\cdots x_m^{\nu_m}$. For if f has more than one term, we get $p_1^{\nu_1}\cdots p_m^{\nu_m} = u = p_1^{\mu_1}\cdots p_n^{\mu_n}$ where $(\nu_1,\cdots,\nu_n)$ and $(\mu_1,\cdots,\mu_m)$ are distinct m-tuples of integers, absurd. Now let $\lambda \in L$ and write $\lambda = f/g$ where f, g are relatively prime. If $\lambda^\sigma = \lambda$, then $fg^\sigma = f^\sigma g$, $g^\sigma = ug$, $f^\sigma = vf$. By the above g and f are monomials: $g = bx_1^{\nu_1}\cdots x_m^{\nu_m}$, $f = bx_1^{\mu_1}\cdots x_m^{\mu_m}$. The relation $\lambda^\sigma = \lambda$ then forces $\mu_i = \nu_i$ for all i and $\lambda = a/b \in \mathbf{Q}$.

REMARK 4.7. (i) Take L, σ, and T as in 4.3. If L is algebraic over k, then the theorem does not apply. For $K = L \otimes k(x_1,\cdots,x_n)$ has dimension zero by 4.4 and being a Noetherian domain is necessarily a field. Thus 4.5 gives no information. In this case, however, classical results [10, pp. 29–30] show that $K[t,t^{-1};\sigma]$ is a principal right ideal domain, not a division ring, and so has both dimensions equal to one. As t is manifestly transcendental over k, $\operatorname{tr.deg.}(D/k) \geqq 1$. A now familiar argument establishes the reverse inequality.

(ii) One can consider more general Ore extensions of the form $T = K[t,t^{-1};\sigma]$ where K, say, is a commutative domain and σ is a k-automorphism of K of infinite order. The first three parts of 4.3 will not hold in this generality, but the formula for transcendence degrees can be saved. For let L be the field of fractions of K and L^σ the fixed field of the extension of σ. Passing to the ring $L[t,t^{-1};\sigma]$ and applying 4.3, we see that $\operatorname{tr.deg.}(D/L^\sigma) = \operatorname{tr.deg.}(L/L^\sigma)$. The additivity of transcendence degree then gives $\operatorname{tr.deg.}(D/k) = \operatorname{tr.deg.}(K/k)$.

(iii) Let L, σ be as in Example 4.6, let $K = L \otimes_{\mathbf{Q}} \mathbf{Q}(y_1,\cdots,y_m)$ and extend σ to K by putting $\sigma(y_i) = y_i$. Arguing by induction on the number of variables, one shows that if $0 \neq I$ is an ideal of $L[y_1,\cdots,y_m]$ with $\sigma(I) \subset I$ then I necessarily contains a nonzero element of the fixed ring $\mathbf{Q}[y_1, \cdots, y_m]$. Thus K has no nontrivial σ-invariant ideals. Let $A = K[t; \sigma]$. By an observation of Jordan [12, p. 93], tA is the unique minimal nonzero prime ideal of A. It follows that A is a Noetherian G-ring (for the definition, see [12, p. 94]) of classical Krull dimension $m + 1$. This shows that a recent result of Amitsur–Small [3] for Noetherian G-rings with polynomial identity fails in the general case.

Let L be a field of characteristic zero, d a nonzero derivation of L with $k = \operatorname{Ker} d$. Let $R = L[y; d]$ be the ring of linear differential operators over L. As a consequence of work in [2], the maximal commutative subalgebras of R all have transcendence degree at most $\operatorname{tr.deg.}(L/k)$. In the following proposition we extend this result to D, the division ring of quotients of R.

THEOREM 4.8. *Let k, L, R, D be as above and let* $\operatorname{tr.deg.}(L/k) = m$ *(m need not be finite but is necessarily nonzero* [6, p. 230]).

(i) $R(x_1, \cdots, x_n)$ *is a central simple $k(x_1, \cdots, x_n)$-algebra.*

(ii) $\operatorname{K dim}[R(x_1, \cdots, x_n)] = \min\{n, m\} = \operatorname{gl. dim}[R(x_1, \cdots, x_n)]$.

(iii) $\operatorname{K dim}[D(x_1, \cdots, x_n)] = \min\{n, m\} = \operatorname{gl. dim}[D(x_1, \cdots, x_n)]$.

(iv) *If $m < \infty$, then* $\operatorname{tr.deg.}(D/k) = m = \operatorname{tr.deg.}(R/k)$.

The proof of 4.8 is very similar to that of 4.3. We provide a sketch but leave the details to the reader. Let $K = L \otimes k(x_1, \cdots, x_n)$ and extend d to K by putting $d(\lambda \otimes f) = \lambda^d \otimes f$. As in 4.3, one shows that $R(x_1, \cdots, x_n) \simeq K[y; d]$. The dimensions of K are given by 4.4. To compute those of $K[y; d]$ one needs to use Hart's theorem [9] in its original form rather than 4.5. Finally, to see that $K[y; d]$ is simple and that Hart's theorem applies use [7, theorem 3.2, p. 43] and argue as in 4.3(i).

The following gives an example of a field satisfying the hypotheses of 4.8 and is adapted from [9, p. 344].

EXAMPLE 4.9. Let F be a field extension of $\mathbf{Q}$ with $[F: \mathbf{Q}] = m$ and let $\{\theta_1, \cdots, \theta_m\}$ be a $\mathbf{Q}$-basis for F. Let $C = F[x_1, \cdots, x_m]$ and let L be the quotient field of C. Put $d(x_i) = \theta_i x_i$ and make d into a derivation of C by using the product rule and distributivity. Then extend d to L by the quotient rule. An argument similar to that of 4.6 shows that $\operatorname{Ker} d = F$.

REMARK 4.10. Let D be a division ring with center k and define $d(D) = \sup_n \{\operatorname{K dim}[D(x_1, \cdots, x_n)]\}$, with $d(D) = \infty$ if the set of integers on the right is unbounded. In light of the above examples, it seems reasonable to ask if the equality $d(D) = \operatorname{tr.deg.}(D/k)$ always holds (we put $\operatorname{tr.deg.}(D/k) = \infty$ if not finite). Though $d(D) \geqq \operatorname{tr.deg.}(D)$ follows from 3.16, establishing the reverse inequality appears extremely difficult. Indeed, if k is arbitrary and D is algebraic over k, determining the Krull dimension of $D(x_1, \cdots, x_n)$ is but another version of a long-standing open problem in ring theory [11, p. 240]. If k is uncountable, Amitsur's theorem [1] settles the degree zero case. The problem for higher transcendence degrees is open, even for uncountable k.

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